Young Children's Knowledge of Three-Dimensional Shapes: Four Case Studies

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This paper provides case studies of the attempts of four young children (a high achieving first-grader, a high achieving kindergarten student and two low achieving first-graders) to solve three tasks involving three-dimensional shapes and their two-dimensional representations. Detailed descriptions of students' solutions are provided; these highlight the appropriateness of challenging geometric tasks for young students. High achieving students spontaneously made more frequent use of formal geometric terms when explaining their actions.

Spatial skills and understandings are an essential component of the primary school mathematics curriculum. For example, authors of the NCTM's Curriculum and Evaluation Standards for School Mathematics (1989) claim that "[s]patial understandings are necessary for interpreting, understanding and appreciating our inherently geometric world" (p.48). The emphasis in the primary school has shifted from naming geometric shapes and memorising geometric facts and vocabulary to that of understanding the properties of geometric shapes, developing spatial sense and using geometry in solving problems. Authors of "A National Statement on Mathematics for Australian Schools" state that "[d]uring the early years of schooling the emphasis should be on exploration, both free and structured, of the children's environment and objects within it"(p.81).

While, in recent times, there has been an increased interest in research on children's spatial thinking, research relating to early geometric knowledge seems to be limited. Wheatley and Cobb (1990) presented a group of first- and second- grade students with shape covering tasks and, based on an analysis of the students' actions and anticipations, identified five levels of spatial construction. Diezmann (1993), in a study involving babies, second grade students and fifth grade students, investigated children's strategies in the interpretation of diagrams of three-dimensional (3D) shapes. She concluded that an effort needs to be made to improve children's diagram literacy. Leeson (1995) studied kindergarten students' spatial construction of three dimensional shapes and found that young children were capable of significant geometrical thinking. Thorpe (1995) found that her investigation of the development of spatial concepts in preschool aged children seemed to support Piaget's finding that young children initially explore topological concepts before exploring Euclidean concepts. Mitchelmore and White (1996) found that most Year 2 students are able to perceive abstract angles in physical situations when both arms of the abstract angle are clearly present.

Owens who has been extensively involved with the assessment of children's spatial concepts (e.g. 1992a; 1992b; 1994) has sought to gain a wider range of children's spatial understanding. She (1992a) devised a group 'pencil and paper' test which assessed children's ability to recognise, reproduce and analyse shapes.

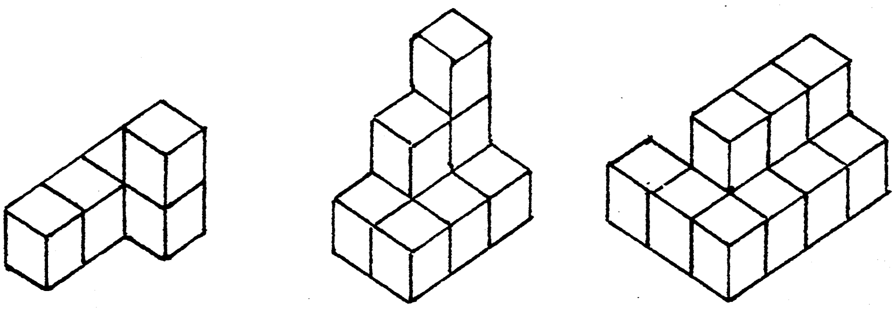
This paper reports on an investigation of young children's spatial knowledge as explored in three tasks involving 3D shapes. It forms part of a project which was undertaken in response to a perceived need to gain a deeper understanding of the spatial and measurement knowledge of students in their first or second years of formal schooling. An important aim of the project is to extend to the assessment of students' early spatial and measurement knowledge, an interview-based approach to the assessment of early arithmetical knowledge. Additional aims of the project include the documentation of students' advancements in spatial and measurement knowledge over the course of a school year and a comparison of students' arithmetical, spatial and measurement knowledge. One of the questions addressed by this study was: What spatial knowledge is possessed by students in Kindergarten or Year 1 (i.e. first-grade) in NSW? The project drew on methodologies typically used in the Maths Recovery project (see Wright, in press). These include individual interview tasks designed to assess a wide range of spatial thinking, and which are problematic for the student. Assessment interviews are routinely videotaped for subsequent analysis.

The project which has been reported by Stewart, Leeson and Wright (1997) involved the development of a space assessment schedule which covered three broad areas of students' spatial knowledge: two dimensional (2D) space, 3D space and position. Within the area of 2D space students' abilities to name, recognise, draw, construct, classify and identify shapes and their understanding of symmetry and tiling were assessed. In the area of 3D space, students were required to sort, classify and name objects and to interpret 2D representations of 3D shapes.

This paper focuses on three spatial tasks (Tasks 15, 16 and 17 of the Space Assessment) in which students' conceptual connections between 2D shapes and 3D shapes are made. For ease of later reference, these will be designated as Tasks X, Y and Z.

In Task X the student was shown a picture of a structure built from five cubes (Figure 1) and asked to build it from a supply of cubes. The student was then shown, in turn, structures made from nine and 12 cubes (Figure 1) and was asked to use cubes to build each.

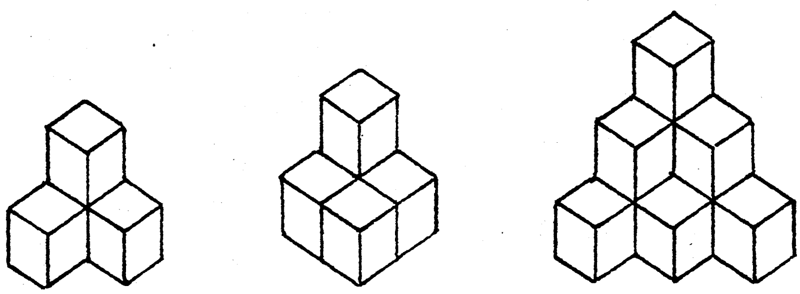
Figure 1. Using cubes to construct shapes.



Task Y assessed whether students could match 3D shapes with their nets. Six solids were displayed in front of the student: a cube, a tetrahedron, a rectangular prism, a triangular prism, a square pyramid and a pentagonal prism. In turn, nets of a cube, triangular prism, square pyramid and tetrahedron were shown to the student who was asked to select the solid which each net could fold to make.

Task Z assessed whether students, by looking at pictures of four, five and ten stacked cubes (Figure 2), were able to determine the number of cubes required to make the structures.

Figure 2. Determining the number of cubes.



Method

Five Kindergarten and five Year 1 students from one school (School A) and five Year 1 students from a second school (School B) participated in the study early in the third school term. As reported by Stewart, Leeson and Wright (1997), it was decided that comparisons could more readily be made if some numerical value could be assigned to each student's performance on the tasks. Analysis of the space assessments involved writing protocols, i.e. descriptions of each student's solutions as well as student's and interviewer's statements. Subsequent analysis involved assigning a numerical value to the student's solution of each task. The value was determined according to task-specific criteria. This lead to an aggregate score for their performance over the space assessment, out of a maximum of 70.

Results

Students' total scores for the space assessments ranged from 66 to 35. In Table 1 scores for the three selected 3D spatial tasks are shown for the highest scoring first-grader, the highest scoring kindergarten student and the two lowest scoring first-graders, respectively. Their arithmetical stages and their total space scores are also shown.

Table 1
Comparison of Arithmetical Stage and Spatial Scores

Name	Arithmetical Stage	Total Space	Score on Spatial Assessment Task X	Task Y	Task Z
Beth	3	66	3	8	4
Isaac	1	43	2	2	0
Michelle	2/3	37	1	6	1
Mark	2	35	1	2	3

The Case of Beth

Beth, aged 7.0 years at the time of assessment, was a student of Year 1 at School A. She obtained the highest aggregate for the space assessment. Only one other child was assessed as being at a higher arithmetical stage.

For Task X, she successfully built the structure having five cubes, by placing three cubes in a row, then one at right angles to the row and finally one in correct position on top. When attempting the structure containing nine cubes, she initially placed the four visible bottom-layer cubes, then she added a tower of two cubes and a second tower of two cubes, each in the correct position. For the third structure she first placed the six visible bottom-layer cubes, then completed the bottom layer and finally added the three cubes of the top layer.

For Task Y, Beth correctly matched the cube with its net and explained that "if you take the top (of the cube) and fold out the sides it would look like that (the net)". She correctly matched the triangular prism with its net, stating, as she pointed to sides of the solid, "there's a square there and there and a triangle there and there and a square on the bottom". The square pyramid was matched with its net because "there's a square in the middle and this (triangle) out there, this out there, this out there and this out there". She also correctly matched the tetrahedron and its net because "it (the solid) has three triangles .... no .... one on the bottom as well".

For Task Z, Beth correctly stated that the first picture contained four cubes because "that one (the one on top) can't stand there without one underneath". She also correctly stated that five and ten cubes, respectively, were required for the second and third pictures.

The Case of Isaac

Isaac was aged 5.4 years at the time of the assessment. He was the highest scoring kindergarten student for the space assessment. Two other kindergarten students were assessed at a higher arithmetical stage (1/2), though it was only marginally higher than his stage.

When building the structure made of five cubes in Task X, Isaac placed four cubes in a row and then placed a fifth on top of the cube at the end of the row. He then checked the diagram and corrected his structure. For the structure comprising nine cubes, he first placed three in a row, then another three beside them to complete the bottom layer; two cubes were placed on top of these but in the wrong position and with a gap between them; then a cube was placed on top of the latter two. When building the structure made of 12 cubes, he initially placed three cubes in a row parallel to the table edge and three in a row perpendicular to these, to form the 'L' shape that was evident in the bottom row of the picture. He then checked the diagram and turned his construction 90 degrees in an anti-clockwise direction. Then he counted the visible cubes of the bottom layer of the diagram and checked his construction. He then added two cubes to the bottom layer and placed one on top. He paused and, when asked if he had finished, re-checked the diagram and then correctly added two more cubes to the second layer.

For Task Y, when presented with the net of the cube, Isaac remarked "That is a cross now; if you folded that one (a square) away it would make one of these (triangular prism)". When asked if he was certain, he then chose the square pyramid; when asked why it would be that one, he responded "because it would". The net of the triangular prism was matched to the tetrahedron because "they (the net) have got triangles and squares in the middle". When the interviewer questioned "This one (tetrahedron) has triangles and squares in the middle?", he re-examined the shape, reflected and then chose the correct solid. When shown the net of the square pyramid, he stated "If you folded these (the triangles) away it would make a square" and then he chose the cube. He chose the triangular prism to match the net for the tetrahedron because, as he pointed to each triangular face of the prism, "that is a triangle and this is a triangle".

Isaac claimed that the first of the pictures for Task Z depicted a structure comprising three cubes. When the interviewer pointed to the top cube in the picture and asked whether it was on the table or in the air, he insisted that it was laying down flat and he maintained that three cubes only were needed. In the same way he maintained that 4 cubes and 6 cubes were required to make the second and third structures.

The Case of Michelle

At the time of interview, Michelle, who was in Year 1 of School B, was aged 6.8 years. Only two of the first-grade students were assessed with an arithmetical stage lower than hers and these were only marginally lower. Her aggregate space assessment score was better than that of only one other first grade student and it was only marginally better.

Michelle successfully constructed the first structure of Task X by placing three cubes in a row, then one at right angles to the row and finally one in the correct position on top. She correctly formed the rectangular-shaped bottom layer of the second structure but incorrectly placed the remaining cubes by placing them on the two end cubes; when the interviewer turned Michelle's structure through 90 degrees to afford her a different view of it, she maintained that it was correct. For the third structure she placed four cubes in a row and a further four in a row at right angles to the former; she then placed three blocks on top of those in the row formed first .

For Task Y she matched the cube with its net because "when you fold that (the net) up you get that (the cube)". The triangular prism was matched with its net because "these two bits (triangles of the net) are them bits (on the solid) and these two (squares of net) are them (on the solid)". The net of the square pyramid was matched with the tetrahedron.

I: Why did you say that one?

M: `cause that one (a triangle of the net) for this one (a triangle of the solid), that one for this and that one for this.

I: (pointing to square of net) And what about this one?

Michelle looked intently at the tetrahedron while turning it around in her hands.

I: Has it got one of those (a square) in it?

M: No!
I: Maybe there's another shape there that does.
Michelle selected the square pyramid.
M: (enthusiastically) This one!
I: You think that one?
M: That bit (square of net) for this one (square of solid).

The net of the tetrahedron was matched to the triangular prism. When offering her explanation, Michelle realised that the solid "don't have a bottom like that" (the triangles of the net) but she was unable to select the matching solid.

For the first structure of Task Z, she initially stated that there were three cubes.
I: Are you sure?
M: Four
I: Four. Why did you change to four?
M: `cause one square (cube) for that one.
I: Oh, I see and where's this one, up in the air or on the table?
M: On the blocks.
I: Are you saying there's another block under this one?
Michelle does not respond.
I: Is there a block under this one or is it on these two blocks?
M: These two.
She then claimed that there were four cubes in the second structure.
I: No hidden ones?
M: No.
Similarly she claimed that the third structure was composed of six cubes.

The Case of Mark

Mark was also from School B and was aged 6.2 years at the time of interview. Of all the first-graders, he was lowest on arithmetical stage and lowest scoring on the space assessment.

For Task X he was unable to build the structure containing five cubes; he placed four cubes in a row and then placed a fifth on the last one of the four. The structure containing nine cubes was successfully completed quite quickly: he formed two rows of three for the bottom layer, then added the two cubes of the second layer and finally the one on top. For the third structure he first made a "L" shape with a row of four cubes at right angles to a further row of three cubes; a cube was placed on each of the latter three; then cubes were placed on the table, on the outer side of the "L", so that there was one next to the corner cube and each of these towers of two.

For Task Y Mark matched the cube with its net because "that there goes there (as he pointed to the squares * in Figure 3) and you fold the wings (squares # of Figure 3) up and it would come into a big square shape like this". The net of the triangular prism was matched with the square pyramid and explained in terms of folding up two squares and both triangles. When presented with the net of the square pyramid he touched each side and pretended to fold it up; he then selected the triangular prism and claimed that "if you fold them two (two of the triangles of the net) it's still got a point coming out there (pointing to the apex of each of the two triangles of the triangular prism)". The net of the tetrahedron was matched to the square pyramid and he claimed that by folding up a triangle of the net "it comes into one of these (the square pyramid)".

Figure 3. Net of cube.

Mark initially stated that there were three cubes in the first picture of Task Z.

I: Where's this one, up in the air or on the desk?

M: In the air

I: Can it sit in the air?

M: It's got a block under it.

He then correctly stated that there were five and ten cubes respectively in the second and third drawings.

Discussion

A common element of Tasks X, Y and Z was interpreting a 2D representation of a 3D shape. There is a perceptual "distance" between a 3D shape and its abstract 2D representation (Parzysz, 1988). When interpreting diagrams of 3D shapes, it is necessary to offset the loss of information caused by the reduction of a geometric shape from 3D to 2D, by creative imagination.

It seems from the case of Beth that we could expect more advanced first grade students to be able to interpret 2D representations of simple 3D structures of the kind used in Tasks X and Z. Further these students should be able to match basic 3D shapes with their nets.

It is noteworthy that Isaac had reasonable success with Task X but failed to obtain any correct answers for Task Z. Perhaps this indicates that he needs physical materials with which to model 3D shapes from their 2D representations (as in Task X), being unable to simply visualise the shapes (as required in Task Z). For Task Y, which also required visualisation, his score was relatively low. From Isaac's case, then, it appears that some of the more advanced kindergarten students may require hands-on materials to help them in their interpretation of 2D representations of 3D shapes.

In an earlier study by Leeson (1995) most of a group of 33 kindergarten students were able to match a net with its associated solid. In that study, however, both nets and solids were made from Polydron interlocking material, whereas in this study 2D drawings of nets were used while solids were made from Polydron material. It seems that the more abstract presentation of the task, as in this study, makes it more difficult for the kindergarten student to make a correct match.

It might have been expected that less advanced first-grade pupils would also need hands-on materials, yet Mark achieved better scores for Task Z than for the other two tasks. Michelle, on the other hand, had best success with Task Y. It appears that less advanced first-graders are unable to consistently offset the perceptual distance between a 3D shape and its 2D representation.

Isaac's response, "because it would", when explaining his reason for choosing a square pyramid to match the net of a cube, reveals that he is operating at a global perceptual level.

Beth and Isaac made more frequent use of formal geometric language than did the two less advanced students. It appears that spontaneous use of more formal geometric language may be a characteristic of the more advanced kindergarten and Year 1 students.

Conclusion

For this study, kindergarten and first grade students were presented with tasks which were quite challenging. There was no indication that any of the students were unnerved or bothered by the prospect of having to solve these tasks. To the contrary, the students appeared to find the tasks interesting and they seemed to enjoy the challenge afforded by the problem solving tasks. This study confirms that the tasks are appropriate for kindergarten and first-grade students.

The tasks might also be suited to problem solving in small group situations. It would be useful to further explore problem solving in early spatial work, in particular, to document across a range of tasks levels of sophistication in students' strategies.

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